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Bulk, surface, and interfacial waves in anisotropic linear elastic solids

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Abstract

The current state of what is known about bulk, surface (Rayleigh), and interfacial (Stoneley) waves in infinite and semi-infinite anisotropic linear elastic solids is surveyed. Features of wave propagation which are not usually presented or even found in treatments involving purely isotropic solids are discussed, and the use of the so-called Stroh formalism for treating certain plane problems of steady wave propagation is reviewed. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

Wave propagation in linear elastic solids has been the subject of investigations by many distinguished researchers in both mathematics and physics since the early part of the nineteenth century. Love (1944), in the Historical Introduction to his classic treatise on the mathematical theory of elasticity, provides an excellent perspective of the early development of the subject, and J.D. Achenbach's book (Achenbach, 1975) covers more modern aspects of the theory of wave propagation in elastic solids; both texts deal almost exclusively with isotropic linear elastic solids. As Achenbach (1975) has noted, "A complete solution of a wave propagation problem involves a considerable amount of mathematical analysis". Indeed, wave propagation problems have provided a beautiful training ground for budding applied mathematicians to sharpen their skills in analysis. This presentation seeks to introduce the reader to the fact that certain aspects of wave propagation in linear elastic solids of quite general anisotropy can be studied in a very simple fashion. To this end we focus our attention on steady wave propagation, which has proved amenable to analyses based on a formalism introduced by Stroh (1962) and for which the

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mathematical complexities associated with transient problems may be avoided. We shall be especially interested in bulk, surface and interfacial waves in elastic media of arbitrary anisotropy.

Three points should be mentioned at the outset. Firstly, the inclusion of anisotropic effects upon the propagation of mechanical disturbances in solids is of more than academic interest, since waves localized near surfaces and interfaces provide excellent probes for non-destructive evaluation of materials containing sub-surface cracks or welds; bulk waves may be used for determining elastic constants experimentally. In addition, an extension to treat steady waves in piezoelectric solids can and has been made, so that wave propagation in modern 'smart' materials may be addressed. Secondly, we remark that there is a large body of work on wave propagation in anisotropic media to be found in the Russian literature; unfortunately, much of the Russian research seems to have been ignored in the West, and it is hoped that in some small way this article may aid in reversing that trend. In particular, the books by Fedorov (1968) [in English], and by Balakirev and Gilinskyi (1982) [in Russian and dedicated to piezoelectric wave phenomena], may be recommended; some of the larger body of Russian literature found only in archival journal form are listed in the bibliographies of journal articles referred to in the body of this work. Finally, the book by Musgrave (1970) and the two volume text by Auld (1973) (the former dealing with crystal acoustics and the latter with acoustic fields and waves in solids) are among the most noteworthy of the books with extensive treatments of anisotropic effects on wave propagation in linear elastic and piezoelectric solids.

2. Isotropic solids

It is generally known that in a homogeneous isotropic linear elastic solid of infinite extent there exist three bulk waves (homogeneous plane waves) associated with each propagation direction **k** (by 'exist' we mean that these waves are solutions to the equations of motion of the elastic solid). One of these waves is a longitudinal wave whose (particle) displacement field is along **k** and whose phase speed is $\sqrt{(\lambda + 2\mu)/\rho}$, where λ and μ are the Lamé constants and ρ is the mass density of the solid. The other two allowable bulk waves are shear waves traveling with phase speed $\sqrt{\mu/\rho}$ (the shear wave speed); the two shear waves are circularly polarized in the plane normal to **k**. If, instead of an infinite medium, we consider an elastic half-space of the same material, it turns out that a so-called 'free surface' wave or Rayleigh wave solution always exists. A surface wave is a steady wave corresponding to a displacement field which has the character of a harmonic wave at the half-space boundary and whose amplitude diminishes exponentially with depth into the solid (more precisely, in an isotropic half-space the Rayleigh wave is composed of two partial waves, each being an inhomogeneous plane wave whose amplitude attenuates exponentially with depth); the free surface or Rayleigh wave is distinguished by the additional feature that it produces no tractions on the half-space boundary. There is only one such wave possible in an isotropic half-space, and its phase speed at the free surface (the Rayleigh wave speed) varies from 0.862 to 0.995 of the shear wave speed, depending on the value of Poisson's ratio ν, when $0 \leq \nu \leq 1/2$. Interfacial or Stoneley waves are waves localized at the interface between two perfectly bonded dissimilar elastic half-spaces, in the sense that the displacement field is a harmonic wave in the interface which suffers an exponentially decreasing amplitude with depth into either medium (more precisely, the displacement field in each isotropic half-space consists of two partial waves, each being an inhomogeneous plane wave whose amplitude attenuates exponentially with depth into the half-space); the associated displacements and tractions are continuous at the interface in the case of perfect bonding. Unlike the Rayleigh wave, which always exists if the half-space is thermodynamically stable, i.e., if its elastic stiffness tensor is positive definite, a Stoneley wave may or may not exist, depending on the elastic constants and mass densities of the two joined half-spaces. When it does exist in the case of two bonded dissimilar isotropic half-spaces, only one such interfacial wave is possible, and its phase speed in the interface is called the Stoneley wave speed; from the theory of Stoneley waves in bonded anisotropic half-spaces (Barnett et al., 1985) it is now known that the Stoneley speed (when the material properties permit a Stoneley wave to exist) is larger than the smaller of the Rayleigh speeds associated with the two half-spaces. We remark that, due to the absence of a characteristic length, the bulk, surface, and interface waves discussed above are dispersionless, a feature that carries over to solids of general anisotropy.

3. Anisotropic solids

An inclusion of anisotropic effects upon wave propagation phenomena in linear elastic solids changes certain features of bulk, surface, and interfacial wave solutions. As it turns out, the discussion of these waves in anisotropic media proceeds best if one begins with bulk waves and concludes with interfacial waves.

3.1. Bulk waves

As in isotropic media of infinite extent, corresponding to any propagation direction $\mathbf{k}$ ($\mathbf{k}$ is real and is called the wave vector, its magnitude k is the wave number, and the unit vector $\mathbf{k}/k$ is called the wave normal) are three bulk waves (homogeneous plane wave solutions) whose displacement fields take the form

$$\mathbf{u} = \mathbf{A}\exp\left(ik\left[\frac{\mathbf{k}}{k}\cdot\mathbf{x} - vt\right]\right),$$

where the real vector $\mathbf{A}$ is the polarization of the wave, v is its phase speed and t is time. The three polarizations and the squares of the three allowable phase speeds are the eigenvectors and eigenvalues, respectively, of the real symmetric matrix

$$\rho^{-1}C_{ijkl}\tau_i\tau_l \ (\tau = \mathbf{k}/k);$$

without the factor ρ^{-1}, this matrix is often called the acoustical tensor or the Christoffel matrix corresponding to the direction τ. For a general direction in a medium of no special crystal symmetry the three bulk waves are neither purely longitudinal nor purely transverse (as in the isotropic case), so that the polarizations $\mathbf{A}$ are neither parallel nor perpendicular to the wave normal; nevertheless, the three polarizations always form (or can be chosen to form) a set of three mutually orthogonal real unit vectors. A wave normal for which two or all three of the allowable phase speeds are identical is called an acoustic axis.

3.2. Slowness surfaces and slowness sections

It proves rather useful to introduce the notion of *slowness* (a concept which, apparently, dates back to W.R. Hamilton), which is the reciprocal of speed (or velocity). For a given wave normal in an anisotropic medium, there exist three slownesses corresponding to bulk wave propagation along the wave normal. Considering all possible directions extending outward from a center or origin, the set of allowable bulk wave slownesses defines a three-sheeted slowness surface (with an outer, medial, and inner sheet). A plane section through the center of the slowness surface produces a centered slowness section with an outer, medial, and inner branch; both the slowness surface and any slowness section are centrosymmetric. Computed slowness sections are given in abundance in Musgrave (1970), and three-dimensional views of various branches of slowness sheets may be found in Abbudi and Barnett (1991), Gundersen (1991), and Wang (1992). For isotropic media one notes that the three slowness sheets are spheres, with the outer two spheres (corresponding to the two shear waves) being coincident; correspondingly, any slowness section consists of three circles, with the outer two coincident.

It turns out that the energy velocity (or Poynting vector or group velocity) associated with a point on a slowness sheet has the direction of the normal direction to the sheet at that point; this direction is essentially the direction of energy flow associated with a wave packet formed from plane waves associated with wave normals in a neighborhood of the particular point on the slowness sheet (volume I of Auld, 1973 provides a beautiful discussion of energy velocity, Poynting vector, group velocity, and their relations to slowness, ray, and wave surfaces). Unlike isotropic media, the direction of energy flow is not usually along the wave normal (the energy flow is not perpendicular to the planes of constant phase). Two other notable features of slowness sections are that:

(a) any straight line in the plane of the section either does not intersect any slowness branch or intersects the slowness branches in a total of two, four, or six points (counting points of tangency as two intersections), and
(b) the inner slowness branch is convex.

3.3. Exceptional bulk waves

A concept which, as we shall discuss later, proves crucial in the analysis of existence of Rayleigh waves in anisotropic half-spaces is that of exceptional bulk waves. A bulk wave is exceptional with respect to the family of parallel planes whose normal is $\mathbf{n}$ if the wave produces no tractions on these planes. Isotropy provides a simple example of an exceptional bulk wave, namely the SH (or shear-horizontally polarized) bulk wave (one may easily verify that in an isotropic medium, any shear wave produces no tractions on the plane containing its wave normal $\mathbf{k}$ and its polarization $\mathbf{A}$, so that $\mathbf{n}=(\mathbf{k} \wedge \mathbf{A})/|\mathbf{k} \wedge \mathbf{A}|$ and $\mathbf{A}\cdot\mathbf{n}=0$. It turns out that $\mathbf{A}\cdot\mathbf{n}=0$ for any exceptional wave in a crystal of arbitrary anisotropy, i.e., the polarization of an exceptional wave lies in the plane with respect to which the wave is exceptional.

One might expect that exceptional bulk waves are limited to materials of high symmetry or to high symmetry directions in crystalline solids. This is *not* the case, and, indeed, in a series of three excellent papers entitled "*Exceptional Waves in Triclinic Crystals: I, II and III*", Alshits and Lothe (1979a, 1979b, 1979c) have formally shown that in *any* anisotropic linear elastic medium the wave normals of the exceptional waves describe *lines* (not necessarily singly connected) on the unit sphere of directions. Alshits and Lothe deduced the intimate relationship between exceptional waves and the problem of degeneracy (the existence of acoustic axes), the manifestation of which is the fact that exceptional waves may be propagated along all acoustic axes (An example of this is found in isotropic solids, for which all directions are acoustic axes due to the equality of the two shear wave speeds, and all shear waves have a plane of exceptionality). Points of so-called conical degeneracy (the local tangent surface to the slowness surface is a cone) are *anchoring* points in the sense that lines of exceptional wave solutions diverge from and end on them on the unit sphere of directions. A corollary discussed by Alshits and Lothe is that acoustic axes are a typical feature of crystals and are not to be regarded as exclusively due to the symmetry elements of the crystal class, i.e., their existence is to be expected, even in triclinic crystals, which are crystals with no associated symmetry elements.

3.4. Stroh's formalism for steady waves

Stroh (1962) presented a rather powerful formalism for treating plane steady problems in anisotropic linear elastic solids; the essence of his method is as follows. Let **m** and **n** be two orthogonal real unit vectors, such that **m**, **n** and $\mathbf{t}=\mathbf{m}\wedge\mathbf{n}$ form a right-hand Cartesian triad; we shall consider problems which are plane in the sense that all fields are independent of $\mathbf{t}\cdot\mathbf{x}$, where **x** is the position vector relative to an origin in the **m**–**n** plane, and which are steady in the sense that they involve uniform motion with a speed v parallel to **m**. Stroh showed that any displacement field of the form

$$\mathbf{u}=\mathbf{A}f(\mathbf{m}\cdot\mathbf{x}+p\mathbf{n}\cdot\mathbf{x}-vt), \tag{1}$$

where t is time and f is an arbitrary function of its argument, satisfies the equations of motion provided that the constant polarization vector **A** and the constant p are properly chosen. For any speed v, there are six allowable (**A**, p) pairs which arise as solutions to the 6 dimensional eigenvalue problem

$$[\mathbf{N}]\begin{bmatrix}\mathbf{A}\\ \mathbf{L}\end{bmatrix}=p\begin{bmatrix}\mathbf{A}\\ \mathbf{L}\end{bmatrix}. \tag{2}$$

The exact form of the 6×6 matrix **N** need not concern us here, save to say that it is real and its elements depend only on the elastic stiffnesses, v, **m** and **n**. The auxiliary Stroh 3-vector **L** is proportional to the traction vector induced by the displacement field **u** on planes normal to **n**, and a general solution to a plane steady problem will involve a linear superposition of the six 'partial' solutions associated with the form of **u** given above. Stroh's method is almost a 'Hamiltonian' formulation of elasticity, which he attributes to the 1957 thesis of I.D.C. Guerney at Cambridge University, although the German literature of the 1920's contains similar formulations.

For static problems ($v=0$) it is easily shown that the allowable p_α ($\alpha=1,2,\ldots,6$) occur as three pairs of complex conjugates (for materials whose strain energy functions satisfy strong convexity, i.e., positive definiteness). This state of affairs persists as v is increased from zero until a speed $\hat{v}$ is reached at which the equations of motion first lose ellipticity. The state at $\hat{v}$ is called the first transonic state, and $\hat{v}$ is called the 'limiting speed'. At the limiting speed, either two, four, or all six Stroh eigenvalues are real (two eigenvalues which, at a lower speed, formed a complex conjugate pair have coalesced into two equal real eigenvalues) and for speeds larger than $\hat{v}$ at least two Stroh eigenvalues are always real. There exists a second (resp., third) transonic state corresponding to the smallest speed above which four (resp., all six) Stroh eigenvalues are always real. For the present purposes, it is the first transonic state that is of interest and we simply use the terminology 'transonic state' unless higher transonic states need be specifically accounted for. The speed regimes $0\le v<\hat{v}$ and $v>\hat{v}$ are referred to as 'subsonic' and 'supersonic', respectively, for purposes of classification only.

The first transonic state has a most useful interpretation in terms of the slowness section in the **m–n** plane (see, for example, Chadwick and Smith, 1977). For a plane steady problem associated with the **m**–**n** plane (uniform speed v parallel to **m**), consider a line parallel to **n** and displaced a distance v^{-1} to the right of the centered slowness section, where $0\le v<\hat{v}$. For $v=0$, such a line does not intersect any of the three slowness branches, since it is displaced (in slowness space) to positive infinity in the direction specified by **m**. As v is increased from zero, this line translates in (parallel to **n**) toward the slowness branches, but still without intersecting any branch, until a speed $v=\hat{v}$ is reached, at which the line first becomes tangent to the outermost slowness branch. Usually, at $v=\hat{v}$ there is only one point of tangency between the line and the outer branch, with no other branches participating in tangential contact (a so-called Type 1 transonic state). There are, however, other possibilities (see Chadwick and Smith, 1977), denoted as Type 2 (tangential contact with both the outer and medial branch at one point; all first transonic states in isotropic media fall in this category), Type 3 (tangential contact with all three branches at a single point), Types 4 or 6 (tangential contact with only the outer branch at two or three points, respectively), or Type 5 (tangential contact with both the outer and medial branch at one point and with only the outer branch at a second point). In addition, confluence of the contact points on the outer sheet (higher order contact) leads to so-called zero-curvature transonic states which shall not concern our discussions. We remark that all these transonic states are possibilities (in the sense that their existence does not violate the strong convexity condition), and that all but the Type 3 transonic state may be constructed using the elastic stiffnesses of *real* materials.

3.5. Limiting bulk waves

Consider a Type 1 transonic state for which, say, $p_1 = p_4 = p = \tan\theta$. With the function $f(z)$ in Eq. (1) taken as $\exp(ikz)$, the field in Eq. (1) corresponds to a homogeneous plane (bulk) wave solution with wave vector $\mathbf{k}$ inclined at an angle θ to $\mathbf{m}$ and with phase speed $\hat{v}\cos\theta$. Such a bulk wave is called a 'limiting bulk wave' or simply a 'limiting wave', because it appears at the limiting speed $v = \hat{v}$ in the Stroh formalism. A Type 1 transonic state admits one limiting wave, Type 2 and Type 4 transonic states admit two limiting waves, and Types 3, 5 and 6 have three limiting waves associated with them. A limiting wave is an *exceptional limiting wave* if it leaves planes normal to $\mathbf{n}$ traction-free, and a transonic state is called an *exceptional transonic state* if all of its limiting waves are exceptional; otherwise the transonic state is called a *normal* transonic state. It may be shown that a Type 1 state may be either normal or exceptional, and that Types 2–6 are always normal transonic states (Barnett and Lothe, 1985). Clearly, in an isotropic medium the associated Type 2 transonic state is normal, since only one limiting exceptional wave can be constructed from the two circularly polarized limiting shear waves.

3.6. Subsonic Rayleigh waves

Consider a linear elastic half-space whose plane boundary has the unit normal $\mathbf{n}$. A surface wave displacement field may be constructed from a superposition of three Stroh partial waves, each of whose amplitudes exhibits exponential decay with depth $\mathbf{n}\cdot\mathbf{x} > 0$ into the half-space. Such a displacement field takes the form

$$\mathbf{u} = \sum_{\alpha=1}^{3} E_\alpha \mathbf{A}_\alpha \exp[ik(\mathbf{m}\cdot\mathbf{x} + p_\alpha \mathbf{n}\cdot\mathbf{x} - vt)], \tag{3}$$

where $\mathbf{m}$ and v are the direction of propagation and the phase speed, respectively, of the surface wave in the half-space boundary, and the E_α are constants selected to satisfy the relevant boundary conditions. In the half-space $\mathbf{n}\cdot\mathbf{x} \geq 0$, we use the partial waves corresponding to the three Stroh eigenvalues with positive imaginary part. Such a steady disturbance is possible at any speed v, provided that we supply the appropriate tractions to the half-space boundary, namely, the tractions derived from the displacement field $\mathbf{u}$. A free surface or Rayleigh wave may be maintained with *vanishing* tractions and, if such a disturbance is possible, it will propagate at the Rayleigh speed v_R.

During the period 1972–1985, a series of papers appeared which ultimately answered what had been an open issue in the theory of surface waves, namely, "Given a linear elastic half-space of arbitrary anisotropy, are there directions in the half-space boundary for which steady subsonic Rayleigh wave propagation is not possible?". In 1956, the Irish mathematical physicist J.L. Synge had considered this problem and had shown that the condition determining the Rayleigh speed corresponded to the vanishing of the determinant of a 3 × 3 complex matrix (which is, in fact, the matrix with elements $L_{i\alpha}$ formed from the auxiliary Stroh eigenvectors corresponding to the three p_α with positive imaginary parts). Synge reasoned that this determinant would have both a real and an imaginary part which are unlikely to vanish simultaneously at *one* particular subsonic speed, so that forbidden directions for Rayleigh wave propagation were likely to be the rule rather than the exception. However, attempts to numerically find forbidden directions for Rayleigh wave propagation were unsuccessful. Stroh (1962) showed that if, instead of considering the 3×3 matrix $\mathbf{L}$, one considered $\mathbf{LL}^T$, the Rayleigh wave speed is determined by the vanishing of the determinant of a purely real symmetric matrix; thus, Synge's original analysis was not as restrictive as he originally believed. Stroh's untimely death shortly after uncovering this result prevented his continuing this line of attack on the existence problem for subsonic Rayleigh waves. Interestingly, Ingebrigtsen and Tonning (1969), unaware of Stroh's work, introduced a formalism strikingly similar to Stroh's in a study of the Rayleigh wave problem; they introduced the notion of a Hermitian surface impedance tensor which was later to prove crucial to the proof of existence of subsonic surface wave modes.

Jens Lothe, who had been an external examiner of Ingebrigtsen's doctoral thesis, saw the connection between the Stroh and the Ingebrigtsen formalisms, and, through his interest in the theory of moving dislocations in anisotropic media, was led to the Rayleigh wave theory (as it then stood) in 1972. Barnett et al. (1973) were able to deduce that the 3×3 real symmetric matrix $\mathbf{B} = 2i\mathbf{LL}^T$ is positive definite at $v = 0$ and has eigenvalues which (in the subsonic regime) are monotonic decreasing functions of v, with two of its eigenvalues vanishing at the Rayleigh speed (if a Rayleigh wave exists); they were able to show that whenever a subsonic Rayleigh wave exists, its speed is unique and it is elliptically polarized. Barnett and Lothe (1974) then studied the asymptotic behavior of the eigenvalues of $\mathbf{B}$ as v approaches the limiting speed, $\hat{v}$ and showed that either a subsonic Rayleigh wave exists or an exceptional limiting wave (a 'surface skimming' bulk wave) exists. The Barnett and Lothe analysis relied on an integral representation for $\mathbf{B}$; their analysis of the behavior of the integral near the limiting speed required more precise reasoning, which was later supplied by Chadwick and Smith (1977). The latter two authors also gave quite precise statements of the existence theorem for subsonic Rayleigh waves. The statement of existence by Chadwick and Smith was not as well-sharpened as possible, and it remained for Barnett and Lothe (1985) to add the necessary precision, using not the $\mathbf{B}$ matrix, but rather the impedance matrix, $\mathbf{Z}$, of Ingebrigtsen and Tonning. The existence theorem may be stated in the following form:

> For a given surface wave geometry in which $\mathbf{m}$ is the direction of propagation of the Rayleigh wave in the free boundary and $\mathbf{n}$ is the unit inner normal to the half-space, a subsonic free surface wave exists unconditionally if the first transonic state is not of Type 1. If the first transonic state is of Type 1 and is normal, a subsonic free surface wave exists. If the first transonic state is of Type 1 and is exceptional (the single limiting bulk wave is exceptional), either a subsonic surface wave exists, or a two-component surface wave propagating at the limiting speed (a so-called transonic surface wave) exists, or no subsonic surface wave exists; hence, for any surface wave geometry either a true subsonic or a transonic free surface wave or an exceptional limiting bulk wave (a surface-skimming bulk wave) always exists.

Which case in the existence theorem pertains in a given surface wave configuration can be determined by computing the eigenvalues of the surface impedance tensor at the limiting speed (Barnett and Lothe, 1985). Barnett et al. (1988) later showed that all four of the above possibilities occur in common metals of cubic symmetry, such as iron; a complete study of surface and bulk waves in materials of cubic symmetry has been made by Chadwick and Smith (1982), and a reasonably complete investigation of surface waves polarized in a plane of mirror (reflection) symmetry has been given by Chadwick (1990) and by Barnett et al. (1991). It should be remarked that although in general one needs to construct a subsonic Rayleigh wave using all three partial waves, there are instances in which only two partial waves are needed (isotropy is such an example, as is the case of a surface wave polarized in a plane of material symmetry); when this is the case we speak of a two-component surface wave. A one-component surface wave propagating at a subsonic speed cannot occur, although this is a possibility in the supersonic regime (Barnett and Chadwick, 1991). A rather complete set of possible partial wave constructions leading to surface waves is delineated in Ting and Barnett (1997).

3.7. Subsonic interfacial waves (Stoneley waves and slip waves)

Subsonic interfacial waves localized at the junction between two perfectly bonded dissimilar half-spaces may be studied in a rather simple fashion once the free surface wave problem has been treated. Essentially one attempts to construct the interfacial wave solution using a linear combination of the three partial waves which decay exponentially with depth into each of the half-spaces, with the requirement that both displacements and tractions be continuous across the interface; in the interfacial wave problem the subsonic regime is the phase speed domain in which there are no real Stroh eigenvalues in *either* half-space. In the subsonic regime the continuity conditions at the interface require that the Stoneley wave speed (when such a wave exists) be determined by the vanishing of the (Hermitian) interfacial impedance tensor, which is the sum of the impedance tensor of one medium and the transpose of the impedance tensor for the other medium (Barnett et al., 1985). Unlike the Rayleigh wave problem, there are forbidden directions and forbidden sectors in the interface for which Stoneley wave propagation is not possible. In fact, for a given interfacial wave geometry there are six possibilities for the behavior of the eigenvalues of the interfacial impedance tensor with phase speed, only two of which admit a true subsonic Stoneley wave (Barnett et al., 1985); thus, allowed Stoneley wave propagation is usually the exception and not the rule. If it exists, the subsonic Stoneley wave travels at a unique speed which must be larger than the smallest Rayleigh wave speed associated with the two half-spaces (Barnett et al., 1985).

If the interface between the two half-spaces is frictionless and cannot support shear tractions, so that interfacial slip (i.e. a discontinuity of tangential displacements at the interface) is allowed, an associated interfacial wave mode is called a slip wave (Murty, 1975 and Barnett et al., 1988). Unlike the subsonic Stoneley wave, there may exist zero, one, or two subsonic slip wave modes for a given interfacial wave geometry; as shown by Barnett et al. (1988), two identical half-spaces of iron in sliding contact admit two subsonic slip wave modes — one at the Rayleigh wave speed and another only infinitesimally below the limiting speed. In the doctoral thesis of L. Wang (1992), the slip wave theory presented by Barnett et al. (1988) was extended by Wang and Lothe with respect to existence of second subsonic slip wave modes.

3.8. Supersonic surface waves and spaces of simple reflection

For speeds between the first and third transonic states in the supersonic regime either only two partial waves or only one partial wave can be available for Rayleigh wave construction in a half-space; in the former case there are, additionally, two bulk wave solutions possible, while there are four bulk wave solutions possible in the latter case. Barnett and Chadwick (1991) have shown that the existence of a one-component surface wave (a free surface wave constructed from only one partial wave) requires that two of the Stroh eigenvalues be real (and the negatives of one another) and that the remaining four eigenvalues occur as a complex conjugate pair of multiplicity two; the degeneracy may be either non-semisimple or semi-simple (the latter situation being pointed out by Wang and Gundersen, 1993), correcting an earlier misconception implied in Barnett and Chadwick, 1991). Thus, supersonic surface waves of either the one-component or the two-component variety may only occur with speeds between the first and the second transonic speeds. An excellent account of the Stroh theory applied to supersonic surface waves (as well as numerical results for real crystals) may be found in the doctoral work of Wang

(1992) and of Gundersen (1991). Additional work on one-component surface waves in materials with symmetry may be found in the papers by Norris (1992) and Chadwick (1992).

Perhaps the most interesting feature of supersonic surface waves was uncovered by Alshits and Lothe (1981), who presented the connection between surface wave theory and the theory of reflection. For a given half-space surface wave geometry (propagation direction **m** and half-space unit inner normal **n**), Alshits and Lothe studied what they termed a state of *simple reflection*, i.e., a situation in which the combination of *one incident bulk wave* and *one reflected bulk wave* leaves the half-space boundary traction-free. They proved that when a supersonic surface wave exists, a state of simple reflection coexists in the same configuration and at the same speed; the converse is not true. Thus, a supersonic surface wave is to be found within the *space of simple reflection* (which, for a given material, is characterized by orientation — **m** and **n** — and speed v). Numerical searches for supersonic surface waves may often be pursued more efficiently by searching first for the spaces of simple reflection, so that the Alshits–Lothe theory has practical as well as fundamental implications.

3.9. Other topics

Space limitations preclude detailed discussions of recent extensions to other topics, including extensions of surface, interfacial, and bulk wave theory to linear piezoelectric solids (see Abbudi and Barnett, 1990; Alshits et al., 1994). Suffice it to say that Stroh's methodology can be extended to coupling between the electric and the elastic field, with the consequence that the augmented theory involves an 8-dimensional eigenvalue problem. The additional field quantities such as electrostatic potential and normal component of electrical induction can lead to the appearance of an 'extra' subsonic Rayleigh wave (the so-called Bluestein–Gulyaev wave) under certain conditions; the piezoelectric Stoneley wave problem has been treated with ease in a fashion quite similar to its purely elastic counterpart (Abbudi and Barnett, 1990; Alshits et al., 1994). The theory of supersonic surface waves and of reflection and refraction in bonded and sliding bimetallic half-spaces is far from complete, although recent advances in treating the latter problems are being reported (but are not yet in print) by Russian investigators. Some intriguing work by L. Wang (1995) on ray surface determination and by C.-Y. Wang and Achenbach (1996) on construction of a solution to the anisotropic Lamb's problem based on a synthesis of Stroh-like solutions also open up problem areas which should prove challenging to any researcher interested in wave propagation in anisotropic media.

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